ASTRON 449, Winter 2019 – Problem Set 1

Due Thu Jan. 17, in class.

REGULAR PROBLEMS:

1. BT2, problem 1.2. a) The luminosity density $j(\mathbf{r})$ of a stellar system is the luminosity per unit volume at position $\mathbf{r}$. Define the surface brightness $I(R)$ as the luminosity per unit area on the sky. For a transparent spherical galaxy (i.e., a galaxy in which light is not attenuated along the line of sight), show that the surface brightness $I(R)$ and luminosity density $j(r)$ are related by the formula

$$I(R) = 2\int_R^\infty dr \frac{r j(r)}{\sqrt{r^2 - R^2}}. \tag{1}$$

Here, r is the 3D radius from the center of the stellar system and R is the radial distance projected on the sky.

b) What is the surface brightness profile of a transparent spherical galaxy with luminosity density $j(r) = j_0(1 + r^2/b^2)^{-5/2}$ (the Plummer model)?

c) Invert equation (1) to obtain

$$j(r) = -\frac{1}{\pi}\int_r^\infty \frac{dR}{\sqrt{R^2 - r^2}} \frac{dI}{dR}. \tag{2}$$

Hint: You can use Abel's transform, discussed in Appendix B.5 of BT2. If you encounter a difficult integral, you can use integral tables and/or a software package like Mathematica to simplify some steps.

d) The Sérsic law,

$$I_{\mathrm{m}}(R) = I(0)\exp\left(-kR^{1/m}\right) = I_{\mathrm{e}}\exp\left\{-b_{\mathrm{m}}[(R/R_{\mathrm{e}})^{1/m} - 1]\right\} \tag{3}$$

is a commonly used approximation to the surface brightness profiles of elliptical galaxies (see BT2, section 1.1.3). Here, I_{e} is the surface brightness at the effective radius[1] and the parameter m is the Sérsic index. Empirically, the Sérsic index is correlated with the luminosity of the elliptical galaxy, luminous ellipticals having $m \approx 6$ and dim ones having $m \approx 2$. The middle of this range, $m = 4$, defines the "de Vaucouleurs" or $R^{1/4}$ law, which is a good approximation to many ellipticals. The functions b_{m} must in general be determined numerically but the fitting formula $b_{\mathrm{m}} = 2m - 0.324$ has a fractional error $\lesssim 0.001$ over the range $1 < m < 10$.

Determine numerically the luminosity density in a spherical galaxy that follows the $R^{1/4}$ surface

[1] The effective radius R_{e} is defined as the radius of the isophote containing half of the total luminosity. Note that R_{e} is measured on the plane of the sky and so is different from the radius of a sphere containing half of the luminosity. Mathematically, $\int_0^{R_{\mathrm{e}}} dR R I(R) = \frac{1}{2}\int_0^\infty dR R I(R)$.

brightness law. Plot $\log_{10}\ j(r)$ versus $\log_{10} r/R_{\rm e}$, where $R_{\rm e}$ is the effective radius.

2. Relaxation by strong vs. weak encounters. Consider a stellar system of size R consisting of N stars each of mass m. In class, we derived an expression for the two-body relaxation time $t_{\rm relax}$ of such a system through the integrated effect of a large number of weak interactions. I.e., we considered encounters with impact parameters $b > b_{90} \equiv 2Gm/v^2$, where v is the characteristic relative velocity between stars.

a) Derive the above expression for b_{90}, defined as the impact parameter within which gravitational interactions are "strong" in the sense that the velocity vector is deflected by an angle $\sim 90°$. An order-of-magnitude derivation is sufficient.

b) Derive the time scale $t_{\rm strong}$ for a star in this system to experience a strong gravitational interaction, i.e. an encounter with impact parameter $b < b_{90}$. Simplify your answer as much as possible.

c) Evaluate the ratio $t_{\rm strong}/t_{\rm relax}$. (Hint: You should be able to express your answer in terms of N only.) When are strong interactions important in changing the velocities of stars, relative to weak encounters?

3. Flat rotation curves. Observations show that many galaxies have nearly constant rotational velocity, $v(r) \approx v_{\rm c} = const$, out to a radius well exceeding the light from the stellar disk.

a) Assuming spherical symmetry, derive an expression for the gravitational potential $\Phi(r)$ of such a galaxy.

b) Using your result for $\Phi(r)$, derive the total matter density profile $\rho(r)$ and an expression for the mass enclosed within radius r, $M(r)$.

c) Of your results for $\Phi(r)$, $\rho(r)$, and $M(r)$, which are valid if the mass is distributed in a disk instead of a sphere? Explain your reasoning for each.

d) For the Milky Way, $v_{\rm c} \approx 220$ km/s. What is the mass enclosed within the solar circle, $r_\odot = 8$ kpc? Compare to the total mass of the Milky Way's dark matter halo, $M_{\rm h} = 10^{12}\ {\rm M}_\odot$.

COMPUTATIONAL PROBLEMS:

Note concerning units: In all your programs this term, treat Newton's constant G as a variable whose value can be modified in the code. By default, we will work in dimensionless units and set $G = 1$.

This first computational problem is very basic with regards to the numerical algorithm but will familiarize you with constructing a full program (including data input, data output, and plotting) in Python.

C1. Evolving test particles in a fixed, analytic potential. Write a Python program that evolves the phase-space coordinates (x, y, z, v_x, v_y, v_z) of a test particle in a prescribed, time-independent analytic potential. Structure your program such that it can be run from the command line using a command of form

```
python integrate_analytic.py x0 y0 z0 vx0 vy0 vz0 pot integr t dt output_file
```

where the first six arguments are the initial phase-space coordinates, `pot` is a string that specifies the potential to use, `integr` is a string that specifies integration algorithm, `t` is the duration of the integration (assumed to begin at time zero), and `dt` is the integration time-step. The last argument, `output_file`, is the path to a file to which the particle trajectory will be output in ASCII format with the following structure

```
t0 x0 y0 z0 vx0 vy0 vz0 Etot0 absL0
t1 x1 y1 z1 vx1 vy1 vz1 Etot1 absL1
...
tN xN yN zN vxN vyN vzN EtotN absLN
```

where `t0`, `t1`, ..., `tN` are times corresponding to the time-steps. The final output time, `tN`, should correspond to the final time `t`. The last two columns contain two diagnostic quantities tracked at each time-step: the (total) specific energy of the test particle ($E = v^2/2 + \phi$) and the magnitude of the specific angular momentum vector relative to the origin ($|\mathbf{L}|$, where $\mathbf{L} = \mathbf{r} \times \mathbf{v}$).

a) For this problem, implement the options `pot=kepler`, corresponding to a Keplerian potential with central mass M (the value of which can be hardcoded, i.e. changed by editing the .py source file), and `integr=euler`, corresponding the most basic Euler integrator:

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{x}_n + \mathbf{v}_n dt \\ \mathbf{v}_{n+1} &= \mathbf{v}_n + \mathbf{a}_n dt. \end{aligned} \tag{4}$$

b) Consider the following initial conditions:

$$\begin{aligned} M &= 1 \\ (\texttt{x0},\ \texttt{y0},\ \texttt{z0},\ \texttt{vx0},\ \texttt{vy0},\ \texttt{vz0}) &= (1.0,\ 0.0,\ 0.0,\ 0.0,\ 1.0,\ 0.0). \end{aligned} \tag{5}$$

Show analytically that this corresponds to a circular orbit and evaluate the orbital period T (in dimensionless units). Evaluate E and L for this orbit. Note that these are conserved quantities in a central force potential.

c) Use your code to evolve a test particle with the Euler integrator and the above initial conditions for 100 orbital periods. Do this for three different choices of the time-step: $dt = 0.1,\ 0.01, 0.001$. Upload a copy of the ASCII output file for the case of $dt = 0.01$ with your code (call this file kepler_euler_dt001.dat).

d) Produce a plot to compare the orbit integrations (for the full 100 orbital periods) obtained using the different time-step choices. Your plot should contain three different panels: the first plotting the evolution of the x, y coordinates of the test particle, the second showing the fractional error in the specific energy of the particle versus time (relative to the exact solution obtained analytically), and the third showing the fractional error in the specific angular momentum versus time.

See the example below for how to structure your plot (you can adjust axis limits, etc., as needed). In each panel, show the values corresponding to the different time-step choices using curves of different color and/or line style.

Note how the errors vary with increasing t and decreasing dt.